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Hamilton's equations and supersymplectic flows on (2,2)-dimensional superspace *

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Hamilton's differential equations are written down in full on the supersymplectic superspace $\mathbb{R}^{2|2}$. They consist of six different sets of equations, one of which is Hamilton's equations on the underlying two-dimensional (q, p) phase space. As for the remaining five, three are algebraic and two are dynamical. Of particular interest is the appearance of a connection-type set of equations for parallel transport with structure group $O(1,1)$. The algebraic equations play a crucial role in proving that the integral flow acts as supersymplectic transformations of $\mathbb{R}^{2|2}$, if and only if it is Hamiltonian. The fact that the supersymplectic transformations depend on odd parameters is fundamental. Finally, the conditions under which the integral flow defines a supergroup action of $\mathbb{R}^{1|1}$ are also given.

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Introduction

The purpose of this paper is to write down in full detail the system of differential equations corresponding to Hamilton's equations in a simple, non-trivial, supersymplectic superspace, $\mathbb{R}^{2|2}$. We follow the approach of ref. [11] to integral flows and superdifferential equations on the one hand, and the general results for finite-dimensional supersymplectic supermanifolds of ref. [4] on the other. Some insight into the structure of the supergroup of supersymplectic transformations is obtained by using the complete set of equations to prove that the integral flow of (super-)Hamilton equations is supersymplectic if and only if it is (super-)Hamiltonian, a generalization to supermanifold theory of the well known result. It is seen along the way that when the full system of equations is solved, the flow results in an element of the supergroup of supersymplectic transformations depending non-trivially on the odd parameter coming from the integration process.

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BACKGROUND

The first reference on the subject of superdifferential equations is due to Shander [13] (we refer the reader to ref. [6] for a concise statement of his results). He proved a rectification theorem for superfields satisfying the generic hypothesis of being *weakly non-degenerate*. His main result states that generically, there are three—essentially distinct—types of superfields. On the other hand, his approach to the ODE problem consists of integrating a superfield by means of one of the rectified forms; the question of which of the three is to be used depends on some set of invariants to be read off from the superfield itself (one such invariant is the Lie self-superbracket, but there are more). In other words, each superfield would dictate which is the best suited way to pose its differential equation, and ultimately to integrate it. In this manner, the integration of superfields is not based on a single universal derivation, but at least on three; the question for non-generic fields was left open still.

The ODE problem was also considered in ref. [8], but from a more universal point of view: ODE's were posed on supermanifolds for every supervector field, and the integration was carried out with the help of a single derivation (∂_t as in the C^∞ category). In this manner, all even superfields could be integrated without special hypotheses of generic type. The authors, however, were unable to integrate certain odd superfields, where even ad hoc techniques within their approach failed to produce an integral flow. Previously, a different formulation of ODE's on supermanifolds was given in ref. [3] following the coalgebra methods of ref. [4]. This was probably the first attempt to explore the Lie theoretic properties of the solution to a superdifferential equation. However, it was limited from the start to even superfields.

In a recent work [11], we addressed the problem of posing and solving differential equations on supermanifolds. Our approach is similar to that of ref. [8], as it uses a single universal derivation to carry out the integration process: $\partial_t + \partial_\tau$, τ being an odd parameter. The advantage is that it produces integral flows not only for homogeneous superfields, but for any superfield satisfying the generic hypotheses. Most recently it has been proved that a unique solution exists *for any superfield* regardless of homogeneity or any other special hypotheses, even of generic type (cf. ref. [10]). *The differential equation* defined by any given supervector field acquires then a precise meaning (cf. refs. [10,11], and §2 below for a brief review).

On the other hand, Hamiltonian systems on supermanifolds have been also considered in the literature. For finite-dimensional supersymplectic supermanifolds, the most comprehensive account is still Kostant's pioneering reference [4]. However, Hamilton's equations were never posed there, perhaps because there was no satisfactory notion at the time of superdifferential equations, nor of integral flows. Nevertheless, Kostant's approach succeeded in deriving many important properties via the algebraic machinery (e.g., cohomological arguments) that may substitute some of the information thrown by the actual integration of the dynamical equations. This approach may be generalized so as to produce infinite-dimensional Hamiltonian systems. The most exhaustive and complete account of this subject can be found in the book of Kupershmidt [5], which has been written with the applications of supersymmetry to classical and quantumn field theories in mind. A transition to finite-dimensional systems from the results in ref. [5] can be obtained via discretization of some equations, but no geometric picture of the flows is made. Instead, proofs of conservation laws are again algebraic, thus omitting any explicit form of an integral flow (i.e., no analog of Stone's theorem is really used). More recently, further work has been done to fill in some of the gaps left in ref. [4]. In fact, the structure of finite-dimensional supersymplectic supermanifolds has now been completely elucidated (cf. refs. [7,12]). Finally, it is worth mentioning that some work on Euler's equations has also been done lately. We refer the reader to ref. [9], where such equations have been written—following the approach of ref. [8]—in general, and in a rather detailed fashion.

1. Supersymplectic structure of $\mathbb{R}^{2|2}$

Our aim is to deal with a specific example in which all the pertinent computations can be performed explicitly, and exactly, so as to actually appreciate the role played by the various pieces of the general structure. Thus, we shall work with the supermanifold $\mathbb{R}^{2|2}$, viewed as

$$\left(\mathbb{R}^2, C^\infty_{\mathbb{R}^2} \otimes \bigwedge(\mathbb{R}^2)^*\right).$$

We shall use greek letters to denote the *odd coordinates* (a basis, $\{\xi, \zeta\}$, of linear functionals on $\mathbb{R}^2$), and the usual (q, p) coordinates for the underlying C^∞ manifold $\mathbb{R}^2$. We shall endow $\mathbb{R}^{2|2}$ with the supersymplectic form

$$\omega = \mathrm{d}q\,\mathrm{d}p + \mathrm{d}\xi\,\mathrm{d}\zeta. \tag{1.1}$$

One notes that $\omega_0 = \mathrm{d}q\,\mathrm{d}p$ is a symplectic form on $\mathbb{R}^2$, and since $\mathrm{d}\xi$ and $\mathrm{d}\zeta$ commute, $\omega_1 = \mathrm{d}\xi\,\mathrm{d}\zeta$ defines a Lorentzian metric on $(\mathbb{R}^2)^*$.

Now, given a superfunction, H, on $\mathbb{R}^{2|2}$, the Hamiltonian supervector field defined by it is completely determined by the condition, $i(X_H)\,\omega = \mathrm{d}H$ (cf. ref. [4]), and it is given in local coordinates by

$$X_H = (\partial_p H)\partial_q - (\partial_q H)\partial_p + (\partial_\zeta H)^\pi \partial_\xi + (\partial_\xi H)^\pi \partial_\zeta, \tag{1.2}$$

where, for any superfunction, $f = f_0 + f_1 \in C^\infty(\mathbb{R}^2) \otimes \bigwedge(\mathbb{R}^2)^*$, $f^\pi = f_0 - f_1$ (the rules followed throughout this work for algebraic manipulations are exactly those given in ref. [4]). If we display the superfunction H in full, we have

$$H = H_0 + H_\xi \xi + H_\zeta \zeta + H_{\xi\zeta}\xi\zeta, \tag{1.3}$$

where H_0, H_ξ, H_ζ, and $H_{\xi\zeta}$ are C^∞ functions on $\mathbb{R}^2$. Thus,

$$(\partial_q H) = (\partial_q H_0) + (\partial_q H_\xi)\xi + (\partial_q H_\zeta)\zeta + (\partial_q H_{\xi\zeta})\xi\zeta,$$
$$(\partial_\xi H) = H_\xi + H_{\xi\zeta}\zeta, \qquad (\partial_\zeta H) = H_\zeta - H_{\xi\zeta}\xi,$$

etcetera. It is now the superdifferential equation defined by the supervector field (1.2) that we want to write down in full.

2. Superdifferential equations in general

We recall from ref. [11] that every supervector field, X, on a superdomain, $\mathcal{M} = (M, \mathcal{A}_M)$ ($\mathcal{A}_M$ being the structural sheaf of superfunctions on the underlying C^∞ manifold M), gives rise to an ODE with prescribed *initial data.* In analogy to the C^∞ theory, a solution is given in terms of the *supertime parameters* $\{t, \tau\}$, which are nothing but a set of (local) coordinates defined (on some neighborhood of $0 \in \mathbb{R}$) on $\mathbb{R}^{1|1}$. To solve the ODE determined by X means to find a supermanifold morphism

$$\Gamma : \mathbb{R}^{1|1} \times \mathcal{M} \to \mathcal{M}, \tag{2.1}$$

such that

$$\mathrm{ev}|_{t=t_0} \circ \widetilde{D} \circ \Gamma^* = \mathrm{ev}|_{t=t_0} \circ \Gamma^* \circ X, \tag{2.2}$$

subject to the initial condition

$$\Gamma \circ (C_0 \times \mathrm{id}) = \mathrm{id}. \tag{2.3}$$

The equality (2.2) is understood as superderivations of the sheaf of superfunctions on $\mathcal{M}$: Γ^* pulls back superfunctions on $\mathcal{M}$ to $\mathbb{R}^{1|1} \times \mathcal{M}$ via Γ; $\widetilde{D}$ is the lift to $\mathbb{R}^{1|1} \times \mathcal{M}$ of a preferred superfield, D, on $\mathbb{R}^{1|1}$. It is defined by the conditions, $\widetilde{D} \circ \pi_1{}^* = \pi_1{}^* \circ D$, and $\widetilde{D} \circ \pi_2{}^* = 0$, π_1 and π_2 being the projections of $\mathbb{R}^{1|1} \times \mathcal{M}$ into the corresponding factors. A few simple arguments (cf. ref. [11]) show that D *must be* chosen as

$$D = \partial_t + \partial_\tau, \tag{2.4}$$

in order to make sense of differential equations, *for any* supervector field. Finally, note that we have used the suggestive symbol "$\mathrm{ev}|_{t=t_0}$" to denote the pull-back algebra map resulting from the inclusion, $C_{t_0} \times \mathrm{id} : \mathcal{M} \to \mathbb{R}^{1|1} \times \mathcal{M}$. Here, C_{t_0} is the constant map defined by the algebra morphism $f \mapsto \tilde{f}(t_0) 1_{\mathcal{M}}$, from superfunctions f on $\mathbb{R}^{1|1}$ to superfunctions on $\mathcal{M}$ (cf. refs. [4,1]).

A fundamental difference with the C^∞ theory is that the integral flow Γ does not always define a Lie supergroup action of $\mathbb{R}^{1|1}$ in the supermanifold. In fact, Lie supergroup actions have been studied in ref. [1], and it has been shown that $\mathbb{R}^{1|1}$ has a naturally defined sum-like morphism,

$$\sigma : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1},$$

which gives to it the structure of an additive supergroup. Now, (2.1) is a Lie supergroup action if, in addition to (2.3), the following equation is satisfied:

$$\Gamma \circ (\sigma \circ (\pi_1 \times \pi_2) \times \pi_3) = \Gamma \circ (\pi_1 \times \Gamma \circ (\pi_2 \times \pi_3)). \tag{2.5}$$

(Here, π_i denotes the projection morphism onto the ith factor of the product $\mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \times \mathcal{M}$). It has been shown in ref. [11] that the supergroup action property for the integral flow is satisfied if and only if the homogeneous components, X_0 and X_1, of the superfield satisfy the following conditions:

$$[X_0, X_1] = 0 \quad \text{and} \quad [X_1, X_1] = 0. \tag{2.6}$$

Now, in combining the morphism Γ with the local problem of actually integrating equation (2.2), a coordinate expression for Γ is needed. Let $\mathcal{M}$ be an (m,n)-dimensional superdomain with coordinates $\{y^i; \theta^\mu\}$. We shall write

$$\Gamma^* y^i = \gamma_0^i + \sum_\nu \gamma_\nu^i p_1^* \tau p_2^* \theta^\nu + \sum_{\mu<\nu} \gamma_{\mu\nu}^i p_2^* \theta^\mu p_2^* \theta^\nu + \cdots, \tag{2.7a}$$

$$\begin{aligned}\Gamma^* \theta^\rho = {} & g_0^\rho p_1^* \tau + \sum_\nu g_\nu^\rho p_2^* \theta^\nu + \sum_{\mu<\nu} g_{\mu\nu}^\rho p_1^* \tau p_2^* \theta^\mu p_2^* \theta^\nu \\ & + \sum_{\lambda<\mu<\nu} g_{\lambda\mu\nu}^\rho p_2^* \theta^\lambda p_2^* \theta^\mu p_2^* \theta^\nu + \cdots,\end{aligned} \tag{2.7b}$$

where p_j is the projection onto the jth factor of the product $\mathbb{R}^{1|1} \times \mathcal{M}$. We shall also write an arbitrary supervector field, X, in the same coordinate patch as

$$X = \sum_i \Big\{ A^i + \sum_\mu A_\nu^i \theta^\nu + \sum_{\mu<\nu} A_{\mu\nu}^i \theta^\mu \theta^\nu + \cdots \Big\} \partial_{y^i}$$

$$+\sum_{\rho}\Big\{B^{\rho}+\sum_{\nu}B^{\rho}_{\nu}\theta^{\nu}+\sum_{\mu<\nu}B^{\rho}_{\mu\nu}\theta^{\mu}\theta^{\nu}+\cdots\Big\}\partial_{\theta^{\rho}}, \tag{2.8}$$

so that finally, the local expressions translate (2.2) into the following series of equations:

$$\begin{aligned}
\partial_t\gamma^i_0 &= A^i\circ\gamma_0, & g^{\rho}_0 &= B^{\rho}\circ\gamma_0,\\
\gamma^i_{\nu} &= \sum_{\lambda} g^{\lambda}_{\nu}A^i_{\lambda}\circ\gamma_0, & \partial_t g^{\rho}_{\nu} &= \sum_{\lambda} g^{\lambda}_{\nu}B^{\rho}_{\lambda}\circ\gamma_0,\\
\partial_t\gamma^i_{\mu\nu} &= \sum_j \gamma^j_{\mu\nu}\partial_{y^j}A^i\circ\gamma_0 & g^{\rho}_{\mu\nu} &= \sum_j \gamma^j_{\mu\nu}\partial_{y^j}B^{\rho}\circ\gamma_0\\
&\quad+\sum_{\eta<\lambda}(g^{\eta}_{\mu}g^{\lambda}_{\nu}-g^{\lambda}_{\mu}g^{\eta}_{\nu})A^i_{\eta\lambda}\circ\gamma_0, & &\quad+\sum_{\eta<\lambda}(g^{\eta}_{\mu}g^{\lambda}_{\nu}-g^{\lambda}_{\mu}g^{\eta}_{\nu})B^{\rho}_{\eta\lambda}\circ\gamma_0,
\end{aligned} \tag{2.9}$$

etc., where use has been made of the fact that for any C^∞ function f on M, $\Gamma^* f$ is given by $f\circ\gamma_0+\sum\gamma^j_{\mu\nu}\partial_{y^j}f\circ\gamma_0\,\theta^{\mu}\theta^{\nu}+\cdots$.

3. Hamilton's superdifferential equations on $\mathbb{R}^{2|2}$

We shall now specialize the general system (2.9) above to the case of the Hamiltonian supervector field X_H, defined by the superfunction H as in (1.3). In doing this, we shall slightly change the notation as follows: The morphism Γ will now be expressed in coordinates as

$$\begin{aligned}
\Gamma^*q &= \gamma^q+\gamma^q_{\xi}\tau\xi+\gamma^q_{\zeta}\tau\zeta+\gamma^q_{\xi\zeta}\xi\zeta,\\
\Gamma^*p &= \gamma^p+\gamma^p_{\xi}\tau\xi+\gamma^p_{\zeta}\tau\zeta+\gamma^p_{\xi\zeta}\xi\zeta,\\
\Gamma^*\xi &= g^{\xi}\tau+g^{\xi}_{\xi}\xi+g^{\xi}_{\zeta}\zeta+g^{\xi}_{\xi\zeta}\tau\xi\zeta,\\
\Gamma^*\zeta &= g^{\zeta}\tau+g^{\zeta}_{\xi}\xi+g^{\zeta}_{\zeta}\zeta+g^{\zeta}_{\xi\zeta}\tau\xi\zeta.
\end{aligned} \tag{3.1}$$

Then, eqs. (2.7) specialize to

$$\begin{pmatrix}\gamma^{q\prime}\\ \gamma^{p\prime}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\widetilde{\gamma}^*\begin{pmatrix}\partial_q H_0\\ \partial_p H_0\end{pmatrix}, \tag{3.2a}$$

$$\begin{pmatrix}g^{\xi}\\ g^{\zeta}\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\widetilde{\gamma}^*\begin{pmatrix}H_{\xi}\\ H_{\zeta}\end{pmatrix}, \tag{3.2b}$$

$$\begin{pmatrix} g_\xi^{\xi\,\prime} & g_\zeta^{\xi\,\prime} \\ g_\xi^{\zeta\,\prime} & g_\zeta^{\zeta\,\prime} \end{pmatrix} = \widetilde{\gamma}^* \begin{pmatrix} H_{\xi\zeta} & 0 \\ 0 & -H_{\xi\zeta} \end{pmatrix} \begin{pmatrix} g_\xi^\xi & g_\zeta^\xi \\ g_\xi^\zeta & g_\zeta^\zeta \end{pmatrix}, \tag{3.2c}$$

$$\begin{pmatrix} \gamma_\xi^q & \gamma_\zeta^q \\ \gamma_\xi^p & \gamma_\zeta^p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \widetilde{\gamma}^* \begin{pmatrix} \partial_q H_\xi & \partial_q H_\zeta \\ \partial_p H_\xi & \partial_p H_\zeta \end{pmatrix} \begin{pmatrix} g_\xi^\xi & g_\zeta^\xi \\ g_\xi^\zeta & g_\zeta^\zeta \end{pmatrix}, \tag{3.2d}$$

$$\begin{pmatrix} \gamma_{\xi\zeta}^{q\,\prime} \\ \gamma_{\xi\zeta}^{p\,\prime} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left\{ \widetilde{\gamma}^* \begin{pmatrix} \partial_{qq}^2 H_0 & \partial_{pq}^2 H_0 \\ \partial_{qp}^2 H_0 & \partial_{pp}^2 H_0 \end{pmatrix} \begin{pmatrix} \gamma_{\xi\zeta}^q \\ \gamma_{\xi\zeta}^p \end{pmatrix} \right. \tag{3.2e}$$

$$\left. + (\det g) \widetilde{\gamma}^* \begin{pmatrix} \partial_q H_{\xi\zeta} \\ \partial_p H_{\xi\zeta} \end{pmatrix} \right\},$$

$$\begin{pmatrix} g_{\xi\zeta}^\xi \\ g_{\xi\zeta}^\zeta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widetilde{\gamma}^* \begin{pmatrix} \partial_q H_\xi & \partial_p H_\xi \\ \partial_q H_\zeta & \partial_p H_\zeta \end{pmatrix} \begin{pmatrix} \gamma_{\xi\zeta}^q \\ \gamma_{\xi\zeta}^p \end{pmatrix}, \tag{3.2f}$$

where

$$(\det g) = \det \begin{pmatrix} g_\xi^\xi & g_\zeta^\xi \\ g_\xi^\zeta & g_\zeta^\zeta \end{pmatrix} \quad \text{and} \quad \widetilde{\gamma} = \begin{pmatrix} \gamma^q \\ \gamma^p \end{pmatrix},$$

and if (a_{ij}) is some matrix of C^∞ functions, $\widetilde{\gamma}^*(a_{ij}) = (a_{ij} \circ \widetilde{\gamma})$, i.e., the pull back under $\widetilde{\gamma}$ of each of the entries.

Note that the usual Hamilton equations (3.2a) are quite independent of the rest of the system. Besides, once the system (3.2a) is solved, most of the expressions in (3.2b)–(3.2f) are determined. For example, eqs. (3.2b) immediately yield the coefficients g^ξ and g^ζ of the flow. Also note that eqs. (3.2c) are of the form $g' g^{-1} = \widetilde{\gamma}^* A$, which, in general, defines parallel transport along $\widetilde{\gamma}$, with respect to the connection (Lie algebra valued one-form) A, on a two-dimensional vector bundle (cf. ref. [2]). In our case, the structure group for this connection is the Lorentz group, $O(1,1)$. After these equations are solved, eqs. (3.2d) are again algebraic and completely determine the matrix on the left hand side. Equations (3.2e) are dynamical, and the classical theorem on existence and uniqueness of solutions for ODE's guarantees a solution of it. That solution may be finally plugged into (3.2f) to completely solve the system uniquely under the initial condition (2.3), which in terms of the coefficients γ and g of the flow reads as follows:

$$\gamma^q(q,p,0) = q, \quad \gamma^p(q,p,0) = p, \quad g_\xi^\xi(q,p,0) = 1, \quad g_\zeta^\zeta(q,p,0) = 1,$$

and all the other coefficient functions take the value zero when evaluated at $t = 0$.

4. Supersymplectic transformations depending on $\mathbb{R}^{1|1}$ parameters

We shall now think of Γ—as given in (3.1)—as a supermanifold morphism,

$$\Gamma_{\{t,\tau\}} : \mathbb{R}^{2|2} \to \mathbb{R}^{2|2},$$

depending on fixed values of the $\mathbb{R}^{1|1}$ parameters $\{t, \tau\}$. We shall impose the condition that $\Gamma_{\{t,\tau\}}$ belong to the supergroup of supersymplectic automorphisms of $\mathbb{R}^{2|2}$, by requiring that

$$\Gamma^*_{\{t,\tau\}}\omega = \omega, \tag{4.1}$$

where the pull back $\Gamma^*_{\{t,\tau\}}\omega$ is given by

$$\Gamma^*_{\{t,\tau\}}(\mathrm{d}q\,\mathrm{d}p + \mathrm{d}\xi\,\mathrm{d}\zeta) = \mathrm{d}\Gamma^*q\,\mathrm{d}\Gamma^*p + \mathrm{d}\Gamma^*\xi\,\mathrm{d}\Gamma^*\zeta, \tag{4.2}$$

with Γ^* being as in (3.1), but treating t and τ as constants. Now, the quadratic structure of ω (in the sense of ref. [12]) is not preserved after pulling it back under an arbitrary morphism. Thus, $\Gamma^*_{\{t,\tau\}}\omega$ will have a decomposition in terms of the various orders of homogeneity of the odd variables with τ involved. In addition to the zeroth order (C^∞) terms, there will be *second order terms* of the form

$$\Omega^2 \otimes \xi\zeta + \Omega^1 \otimes \big((\xi + \zeta)(\mathrm{d}\xi + \mathrm{d}\zeta)\big) + \Omega^0 \otimes \mathrm{d}\xi\,\mathrm{d}\zeta, \tag{4.3a}$$

as well as of the form

$$\Omega^2 \otimes \tau(\xi + \zeta) + \Omega^1 \otimes \tau(\mathrm{d}\xi + \mathrm{d}\zeta), \tag{4.3b}$$

and finally, *fourth order terms* of the form

$$\Omega^1 \otimes \tau\xi\zeta(\mathrm{d}\xi + \mathrm{d}\zeta) + \Omega^0 \otimes \tau(\xi + \zeta)\big((\mathrm{d}\xi)^2 + (\mathrm{d}\zeta)^2 + \mathrm{d}\xi\,\mathrm{d}\zeta\big), \tag{4.3c}$$

with Ω^k standing for the differential (C^∞) k-forms defined on the (q, p) space, $\mathbb{R}^2$. On the other hand, the original supersymplectic form has only the zeroth order term $\mathrm{d}q\,\mathrm{d}p$ and the second order term $\mathrm{d}\xi\,\mathrm{d}\zeta$. By equating the C^∞ coefficients on both sides of (4.1), one obtains a number of equations among the γ's and the g's. Not all of these are independent, and it is a straightforward matter to verify that what remains is the following set of conditions:

$$\mathrm{d}\gamma^q\,\mathrm{d}\gamma^p = \mathrm{d}q\,\mathrm{d}p, \tag{4.4a}$$

$$g^\xi_\xi g^\zeta_\xi = 0, \quad g^\xi_\zeta g^\zeta_\zeta = 0, \quad g^\xi_\xi g^\zeta_\zeta + g^\zeta_\xi g^\xi_\zeta = 1, \tag{4.4b}$$

$$\gamma^q_{\xi\zeta}\,\mathrm{d}\gamma^p - \gamma^p_{\xi\zeta}\,\mathrm{d}\gamma^q + g^\xi_\xi\,\mathrm{d}g^\zeta_\zeta + g^\xi_\zeta\,\mathrm{d}g^\zeta_\xi = 0, \tag{4.4c}$$

$$\gamma^q_\xi\,\mathrm{d}\gamma^p - \gamma^p_\xi\,\mathrm{d}\gamma^q - (g^\xi_\xi\,\mathrm{d}g^\zeta + g^\zeta_\xi\,\mathrm{d}g^\xi) = 0, \tag{4.4d}$$

$$\gamma^q_\zeta\,\mathrm{d}\gamma^p - \gamma^p_\zeta\,\mathrm{d}\gamma^q - (g^\xi_\zeta\,\mathrm{d}g^\zeta + g^\zeta_\zeta\,\mathrm{d}g^\xi) = 0, \tag{4.4e}$$

$$\gamma^p_{\xi\zeta}\gamma^q_\xi - \gamma^q_{\xi\zeta}\gamma^p_\xi - (g^\xi_\xi g^\zeta_{\xi\zeta} + g^\zeta_\xi g^\xi_{\xi\zeta}) = 0, \tag{4.4f}$$

$$\gamma^p_{\xi\zeta}\gamma^q_\zeta - \gamma^q_{\xi\zeta}\gamma^p_\zeta - (g^\xi_\zeta g^\zeta_{\xi\zeta} + g^\zeta_\zeta g^\xi_{\xi\zeta}) = 0. \tag{4.4g}$$

Now, the relation in (4.4a) says that the underlying flow $\widetilde{\gamma} = (\gamma^q, \gamma^p)$ is symplectic. The conditions in (4.4b) express the fact that

$$\begin{pmatrix} g^\xi_\xi & g^\xi_\zeta \\ g^\zeta_\xi & g^\zeta_\zeta \end{pmatrix}^{\mathrm{t}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g^\xi_\xi & g^\xi_\zeta \\ g^\zeta_\xi & g^\zeta_\zeta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

in other words, that the matrix valued function g takes values in the group $\mathrm{O}(1,1)$. Now, eqs. (4.4d) and (4.4e) may be written as equations of constraint by defining first the vector fields

$$X_\xi = \gamma^q_\xi \partial_q + \gamma^p_\xi \partial_p,$$

$$X_\zeta = \gamma^q_\zeta \partial_q + \gamma^p_\zeta \partial_p,$$

and noting that $i_X \mathrm{d}\gamma^q\,\mathrm{d}\gamma^p = i_X \mathrm{d}q\,\mathrm{d}p = i_X \omega_0$, in view of (4.4a). Therefore, (4.4d) and (4.4e) are equivalent to

$$\gamma^q_\xi\,\mathrm{d}p - \gamma^p_\xi\,\mathrm{d}q - (g^\xi_\xi\,\mathrm{d}g^\zeta + g^\zeta_\xi\,\mathrm{d}g^\xi) = 0, \tag{4.4d'}$$

$$\gamma^q_\zeta\,\mathrm{d}p - \gamma^p_\zeta\,\mathrm{d}q - (g^\xi_\zeta\,\mathrm{d}g^\zeta + g^\zeta_\zeta\,\mathrm{d}g^\xi) = 0, \tag{4.4e'}$$

respectively. Here is now our main result:

Theorem. *Let Γ be given by* (3.1). *Suppose Γ satisfies Hamilton's superdifferential equations* (3.2a)–(3.2f). *Then, for each pair, $\{t, \tau\}$, of fixed values of the $\mathbb{R}^{1|1}$ integration parameters, the flow is supersymplectic; i.e., $\Gamma^*_{\{t,\tau\}}\omega = \omega$. Conversely, if the flow is (locally) supersymplectic, then it is the solution to Hamilton's superdifferential equations, for some super-Hamiltonian $H = H_0 + H_\xi \xi + H_\zeta \zeta + H_{\xi\zeta}\xi\zeta$.*

Proof. We have to check first that the solution coefficients, the γ's and g's of the system (3.2a)–(3.2f), satisfy the conditions (4.4a)–(4.4f) above. Now, it is a well known classical result that (4.4a) follows from (3.2a). One also notes that (4.4b) follows from (3.2c), as the latter says that $g'g^{-1}$ is a function that takes values in the Lie algebra of $O(1,1)$. Now, (4.4d) and (4.4e) follow from (3.2b) and (3.2d) by noting that (3.2b) implies that

$$\begin{pmatrix} \partial_q H_\xi & \partial_q H_\zeta \\ \partial_p H_\xi & \partial_p H_\zeta \end{pmatrix} = \begin{pmatrix} \partial_q g^\zeta & \partial_q g^\xi \\ \partial_p g^\zeta & \partial_p g^\xi \end{pmatrix}.$$

On the other hand, (3.2d) implies that

$$\begin{pmatrix} \partial_q H_\xi & \partial_q H_\zeta \\ \partial_p H_\xi & \partial_p H_\zeta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma^q_\xi & \gamma^q_\zeta \\ \gamma^p_\xi & \gamma^p_\zeta \end{pmatrix} \begin{pmatrix} g^\xi_\xi & g^\xi_\zeta \\ g^\zeta_\xi & g^\zeta_\zeta \end{pmatrix}^{-1}.$$

Therefore, one obtains

$$\mathrm{d}g^\xi = \frac{1}{\det g}\left((g^\xi_\zeta \gamma^p_\xi - g^\xi_\xi \gamma^p_\zeta)\,\mathrm{d}q + (g^\xi_\xi \gamma^q_\zeta - g^\xi_\zeta \gamma^q_\xi)\,\mathrm{d}q\right),$$

$$\mathrm{d}g^\zeta = \frac{1}{\det g}\left((g^\zeta_\xi \gamma^p_\zeta - g^\zeta_\zeta \gamma^p_\xi)\,\mathrm{d}q + (g^\zeta_\zeta \gamma^q_\xi - g^\zeta_\xi \gamma^q_\zeta)\,\mathrm{d}q\right),$$

from which (4.4d) and (4.4e) are obtained after taking the appropriate C^∞ linear combinations of the left hand sides. Similarly, (4.4f) and (4.4g) follow from (3.2d) and (3.2f). In fact, from (3.2d) one obtains

$$\begin{pmatrix} \partial_q H_\xi & \partial_p H_\xi \\ \partial_q H_\zeta & \partial_p H_\zeta \end{pmatrix} = \frac{1}{\det g} \begin{pmatrix} g^\zeta_\zeta & -g^\zeta_\xi \\ -g^\xi_\zeta & g^\xi_\xi \end{pmatrix} \begin{pmatrix} \gamma^q_\xi & \gamma^p_\xi \\ \gamma^q_\zeta & \gamma^p_\zeta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Plugging this matrix into (3.2f), we obtain

$$\begin{pmatrix} g^\xi_\xi & g^\zeta_\xi \\ g^\xi_\zeta & g^\zeta_\zeta \end{pmatrix} \begin{pmatrix} g^\zeta_{\xi\zeta} \\ g^\xi_{\xi\zeta} \end{pmatrix} = \begin{pmatrix} \gamma^q_\xi & \gamma^p_\xi \\ \gamma^q_\zeta & \gamma^p_\zeta \end{pmatrix} \begin{pmatrix} \gamma^p_{\xi\zeta} \\ -\gamma^q_{\xi\zeta} \end{pmatrix},$$

as required.

Conversely, assume (4.4a)–(4.4f) hold true. Then (4.4a) already implies—by the well known classical result—that there is a locally defined Hamiltonian function, H_0, satisfying eqs. (3.2a). Equations (4.4b) imply, as we noted before, that the matrix valued function

$$g = \begin{pmatrix} g^\xi_\xi & g^\xi_\zeta \\ g^\zeta_\xi & g^\zeta_\zeta \end{pmatrix}$$

takes values in O(1, 1). In particular, $g'g^{-1}$ takes values in the Lie algebra of O(1, 1), and therefore $H_{\xi\zeta}$ is defined through (3.2c). Now, if we define H_ξ and H_ζ by means of (3.2b), then (3.2d) holds true if and only if (4.4d) and (4.4e) are satisfied [the very same argument used to deduce (4.4d) and (4.4e) from (3.2b) and (3.2d) applies here]. Similarly, with (3.2b) defined, (3.2f) holds if and only if (4.4f) and (4.4g) are satisfied.

Finally, let us prove the one remaining case, namely, that eq. (4.4c) holds if and only if (3.2e) does. We first note that the Hessian appearing in (3.2e) can be replaced; indeed, from (3.2a) one obtains

$$\partial_t \begin{pmatrix} \partial_q \gamma^q & \partial_p \gamma^q \\ \partial_q \gamma^p & \partial_p \gamma^p \end{pmatrix} = \tilde{\gamma}^* \begin{pmatrix} \partial^2_{qp} H_0 & \partial^2_{pp} H_0 \\ -\partial^2_{qq} H_0 & -\partial^2_{pq} H_0 \end{pmatrix} \begin{pmatrix} \partial_q \gamma^q & \partial_p \gamma^q \\ \partial_q \gamma^p & \partial_p \gamma^p \end{pmatrix}.$$

Now, $\tilde{\gamma}$ is a symplectomorphism, and hence

$$\begin{pmatrix} \partial_q \gamma^q & \partial_p \gamma^q \\ \partial_q \gamma^p & \partial_p \gamma^p \end{pmatrix}^{-1} = \begin{pmatrix} \partial_p \gamma^p & -\partial_p \gamma^q \\ -\partial_q \gamma^p & \partial_q \gamma^q \end{pmatrix}.$$

In particular,

$$\begin{aligned} \tilde{\gamma}^* & \begin{pmatrix} \partial^2_{qp} H_0 & \partial^2_{pp} H_0 \\ -\partial^2_{qq} H_0 & -\partial^2_{pq} H_0 \end{pmatrix} \\ &= \partial_t \begin{pmatrix} \partial_q \gamma^q & \partial_p \gamma^q \\ \partial_q \gamma^p & \partial_p \gamma^p \end{pmatrix} \begin{pmatrix} \partial_p \gamma^p & -\partial_p \gamma^q \\ -\partial_q \gamma^p & \partial_q \gamma^q \end{pmatrix} \\ &= - \begin{pmatrix} \partial_q \gamma^q & \partial_p \gamma^q \\ \partial_q \gamma^p & \partial_p \gamma^p \end{pmatrix} \partial_t \begin{pmatrix} \partial_p \gamma^p & -\partial_p \gamma^q \\ -\partial_q \gamma^p & \partial_q \gamma^q \end{pmatrix}. \end{aligned}$$

Thus, we may rewrite (3.2e) as

$$\begin{aligned} \partial_t & \left\{ \begin{pmatrix} \partial_p \gamma^p & -\partial_p \gamma^q \\ -\partial_q \gamma^p & \partial_q \gamma^q \end{pmatrix} \begin{pmatrix} \gamma^q_{\xi\zeta} \\ \gamma^p_{\xi\zeta} \end{pmatrix} \right\} \\ &= \det g \begin{pmatrix} \partial_p \gamma^p & -\partial_p \gamma^q \\ -\partial_q \gamma^p & \partial_q \gamma^q \end{pmatrix} \tilde{\gamma}^* \begin{pmatrix} \partial_p H_{\xi\zeta} \\ -\partial_q H_{\xi\zeta} \end{pmatrix}. \end{aligned}$$

On the other hand, using (3.2c), one obtains

$$\partial_t(g_\xi^\xi \partial_q g_\zeta^\zeta) = -(\partial_q \gamma^q \tilde{\gamma}^* \partial_q H_{\xi\zeta} + \partial_q \gamma^p \tilde{\gamma}^* \partial_p H_{\xi\zeta}) g_\xi^\xi g_\zeta^\zeta,$$
$$\partial_t(g_\zeta^\xi \partial_p g_\xi^\zeta) = -(\partial_p \gamma^q \tilde{\gamma}^* \partial_q H_{\xi\zeta} + \partial_p \gamma^p \tilde{\gamma}^* \partial_p H_{\xi\zeta}) g_\zeta^\xi g_\xi^\zeta.$$

So finally, (3.2e) is reduced to

$$\partial_t \left\{ \begin{pmatrix} \partial_p \gamma^p & -\partial_p \gamma^q \\ -\partial_q \gamma^p & \partial_q \gamma^q \end{pmatrix} \begin{pmatrix} \gamma^q_{\xi\zeta} \\ \gamma^p_{\xi\zeta} \end{pmatrix} + \begin{pmatrix} g_\xi^\xi \partial_q g_\zeta^\zeta \\ g_\zeta^\xi \partial_p g_\xi^\zeta \end{pmatrix} \right\} = 0,$$

where use has been made of eqs. (4.4b) above. Using this version of eq. (3.2e), (4.4c) follows, and conversely, (4.4c) makes this version of (3.2e) to hold true, and the proof is complete. □

Remark. Actually, the theorem is true in general, and not only for $\mathbb{R}^{2|2}$. The argument, however, is not completely elementary. One must define the notion of Lie derivative first, in a manner that links it to the flow Γ on the one hand, and to the algebraic property, $L_X = i(X) \circ \mathrm{d} + \mathrm{d} \circ i(X)$, on the other. In doing so, the supergroup action property of the flow should be avoided. The natural definition is [cf. eq. (3.2)]: for any superform ω,

$$L_X \omega = \mathrm{ev}|_{t=0} \circ \widetilde{D} \circ \Gamma^*_{\{t,\tau\}} \omega.$$

When this is done, and the equivalence with the algebraic definition is settled, the classical well known proof runs well for supermanifolds. Our point in working out all the details on the $\mathbb{R}^{2|2}$ case (without Lie derivatives) was to elucidate the role played by the various coefficients of the flow.

5. Conditions for the super-Hamiltonian flow to define an $\mathbb{R}^{1|1}$ supergroup action

We shall close our discussion by finding out what are the conditions on the super-Hamiltonian, H, for its flow Γ to define a supergroup action of $\mathbb{R}^{1|1}$ on $\mathbb{R}^{2|2}$. As mentioned before, if we are given a non-singular, non-degenerate, supervector field, X, the necessary and sufficient conditions for this to happen are (cf. ref. [11] for a proof)

$$[X_0, X_1] = X_0 X_1 - X_1 X_0 = 0,$$
$$[X_1, X_1] = 2 X_1 X_1 = 0, \tag{5.1}$$

where X_0 and X_1 are the homogeneous components of X, and $[\cdot,\cdot]$ is the supercommutator of supervector fields. Now, let X_H be given by (1.2), and H as in (1.3). We shall write $X_H = X_0 + X_1$, where

$$\begin{aligned} X_0 =& (\partial_p H_0 + \partial_p H_{\xi\zeta}\xi\zeta)\partial_q - (\partial_q H_0 + \partial_q H_{\xi\zeta}\xi\zeta)\partial_p \\ & - H_{\xi\zeta}\xi\partial_\xi + H_{\xi\zeta}\zeta\partial_\zeta, \\ X_1 =& (\partial_p H_\xi\xi + \partial_p H_\zeta\zeta)\partial_q - (\partial_q H_\xi\xi + \partial_q H_\zeta\zeta)\partial_p \\ & + H_\zeta\partial_\xi + H_\xi\partial_\zeta. \end{aligned} \tag{5.2}$$

Now, a straightforward computation leads to

$$\begin{aligned} [X_0, X_0] =& \Big\{(\partial_p\{H_0, H_\xi\} + H_\xi\partial_p H_{\xi\zeta} - H_{\xi\zeta}\partial_p H_\xi)\xi \\ & + (\partial_p\{H_0, H_\zeta\} - H_\zeta\partial_p H_{\xi\zeta} + H_{\xi\zeta}\partial_p H_\zeta)\zeta\Big\}\partial_q \\ & - \Big\{(\partial_q\{H_0, H_\xi\} + H_\xi\partial_q H_{\xi\zeta} - H_{\xi\zeta}\partial_q H_\xi)\xi \\ & + (\partial_q\{H_0, H_\zeta\} - H_\zeta\partial_q H_{\xi\zeta} + H_{\xi\zeta}\partial_q H_\zeta)\zeta\Big\}\partial_p \\ & + \Big\{\{H_0, H_\zeta\} + H_{\xi\zeta}H_\zeta + 2\{H_{\xi\zeta}, H_\zeta\}\xi\zeta\Big\}\partial_\xi \\ & + \Big\{\{H_0, H_\xi\} - H_{\xi\zeta}H_\xi + 2\{H_{\xi\zeta}, H_\xi\}\xi\zeta\Big\}\partial_\zeta, \\ \tfrac{1}{2}[X_1, X_1] =& \Big\{\partial_p(H_\xi H_\zeta) + \partial_p\{H_\xi, H_\zeta\}\xi\zeta\Big\}\partial_q \\ & - \Big\{\partial_q(H_\xi H_\zeta) + \partial_q\{H_\xi, H_\zeta\}\xi\zeta\Big\}\partial_q \\ & + \{H_\xi, H_\zeta\}\xi\partial_\xi - \{H_\xi, H_\zeta\}\zeta\partial_\zeta, \end{aligned} \tag{5.3}$$

where $\{F, G\}$ denotes the Poisson bracket of the C^∞ functions F and G. From these expressions it is not difficult to conclude that

$$\begin{aligned} [X_0, X_1] = 0 \quad &\Longleftrightarrow \quad \begin{pmatrix} 0 & \partial_p H_{\xi\zeta} \\ \partial_q H_{\xi\zeta} & 0 \end{pmatrix}\begin{pmatrix} H_\xi & -H_\zeta \\ H_\zeta & H_\xi \end{pmatrix} = 0, \\ [X_1, X_1] = 0 \quad &\Longleftrightarrow \quad H_\xi H_\zeta = \text{const}. \end{aligned} \tag{5.4}$$

Note that an *even super-Hamiltonian* (i.e., one of the form $H = H_0 + H_{\xi\zeta}\xi\zeta$) always yields a field satisfying these $\mathbb{R}^{1|1}$ *supergroup action conditions*. On the other hand, *odd super-Hamiltonians* (of the form $H = H_\xi\xi + H_\zeta\zeta$) necessarily require $H_\xi H_\zeta$ to be a constant function in order to enjoy the same property.

Finally, non-homogeneous super-Hamiltonians further require $H_{\xi\zeta}$ to be a constant function.

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